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ON THE SYLLOGISM I

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ON THE SILLOGISM. I

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INTRODUCTION

The present paper is the first of a series devoted to the theory of the syllogism. In this series, we plan to study topics such as the following: 1) Two algebraic systems related to the Aristotelian syllogistic; 2) Some basic properties of two formalizations of the categorical syllogistic, one of which is essentially Corcoran's (see Corcoran [1972], [1973], [1974a] and [1974b]); 3) The doctrine of the quantified predicate, and certain logical systems connected with Hamilton's conception of the syllogistic, as presented in his *New Analytics* (see Hamilton [1867]).

In this paper, we construct two systems appropriate to the study of categorical propositions by means of algebraic operations and relations. In our systems, predicates are quantified in the spirit of Boole's well-known method of indeterminate coefficients. We also consider two formulations of Aristotle's syllogistic, and then investigate the four systems, treating their interrelations, fundamental properties, and semantical counterparts.

Our results are mainly concerned with natural deductive formulations of the syllogistic systems considered. As Corcoran and others have shown, this method of presentation captures better the gist of the syllogism, at least in the Aristotelian case. However, our exposition can be adapted to other methods of formalization, for instance that of Lukasiewicz (see Lukasiewicz [1957]).

To give an idea of our programme, let us call *positive* the categorical propositions of the forms A and I, and *negative* those of the forms E and O. If Γ is a set of categorical propositions, we denote by Γ^+ the set of positive propositions in Γ. We show, for example, that in Corcoran's systematization of the syllogism, if Γ is a consistent set of propositions, α is a positive proposition, and $\Gamma \vdash \alpha$, then $\Gamma^+ \vdash \alpha$. Furthermore, if β is a negative proposition and $\Gamma \vdash \beta$, then there is a negative proposition $\gamma \in \Gamma$, such that $\Gamma^+, \gamma \vdash \beta$. Another important point: Corcoran's system contains both *direct* and *indirect* deductions, i.e. it includes, in a certain sense (see Corcoran [1974a], pp. 116-117), the rule of reductio ad absurdum. In order to obtain a more perfect understanding of the formal interplay of the rules employed, we formulate a *direct* syllogistic system, equivalent to Corcoran's.

Soundness and completeness theorems are proved for all systems introduced, relative to their intended semantics.

The rules of our direct version of the syllogistic which may be applied in a deduction from a consistent set of formulas are just those equivalent to some of the traditional rules for immediate inferences and common syllogisms. Therefore, the conpleteness theorem which is proved for this version is the formal vindication of a belief shared by most of the traditional logicians: that any valid inference of a categorical statement from a (consistent) set of categorical premisses can be reduced to a chain of immediate inferences and common syllogisms (cf. Aristotle, *Prior Analytics*, I, 23 and *La Logique de Port Royal*, III, 1).

Most of the results considered here will be employed and exploited in the forthcoming papers of the series.

1. THE ALGEBRAIC LANGUAGE $\mathcal{L}$

The *vocabulary* of $\mathcal{L}$ is composed of the following symbols: 1) an infinitely denumerable set of *simple terms*; 2) a set of *simple functors* (the letters "P" and "Q" with numerical subscripts); 3) the symbols "=" and "$\|$". The *terms*, *functors*, and *formulas* of $\mathcal{L}$ are defined by the following rules:

1) Every simple term is a term; 2) if a is a term and P is a simple functor, then Pa is a term; 3) if a and b are terms, then $a = b$ is a (*positive*) formula and $a \| b$ is a (*negative*) formula; 4) if P is a finite sequence of n simple functors ($n \geq 0$), P is a functor; when $n = 0$, P is the empty functor.

Unless explicit mention is made to the contrary, the last small Latin letters will be employed as variables for simple terms, and the first small Latin letters will be used as variables for terms in general, capital Latin letters will stand as variables for functors in general and, finally, small Greek letters will be used as variables for formulas.

A formula of form $Pa = Pb$ is called an *instance* of the formula $a = b$. If α is an instance of β and β is not an instance of any other formula, then β is said to be the *origin* of α. It is clear that any formula has one and only one origin.

The formula $a = b$ is called the *converse* of $b = a$.

Let Γ denote a set of formulas; then Γ^{*} will denote the set of positive formulas which belong to Γ.

2. INTERPRETATIONS OF $\mathcal{L}$

A function F is said to be *restrictive on a set* U if and only if: i) U is not empty; ii) the domain of F is the set of all nonempty subsets of U; iii) for every x in the domain of F, $F(x) \subseteq x$ and $F(x)$ is not empty.

An interpretation of $\mathcal{L}$ is an ordered pair $\langle \mathcal{U}, \mathfrak{I} \rangle$, where $\mathcal{U}$ is an nonempty set and $\mathfrak{I}$ is a function which associates with each simple term of $\mathcal{L}$ a nonempty subset of $\mathcal{U}$, and to each simple functor of $\mathcal{L}$ a restrictive function on $\mathcal{U}$.

Let I be an interpretation $\langle \mathcal{U}, \mathfrak{I} \rangle$ of $\mathcal{L}$. We define the value $I(z)$ of z according to I, the value $I(P)$ of the simple functor P according to I, the value $I(a)$ of the term a according to I, and the value $I(\alpha)$ of a formula α according to I, respectively, as follows: 1) $I(z) = \mathfrak{I}(z)$; 2) $I(P) = \mathfrak{I}(P)$; 3) $I(Pa) = I(P)(I(a))$; 4) $I(a{=}b) = 1$ when $I(a) = I(b)$; $I(a{=}b) = 0$ otherwise; 5) $I(a \parallel b) = 1$ when $I(a) \cap I(b) = \phi$; $I(a \parallel b) = 0$ otherwise.

We define the terms *model* and *implication* as usual.

3. THE STRONG ALGEBRAIC SYSTEM

We define the *strong algebraic system* as follows: its language is $\mathcal{L}$ and its axioms and rules are those listed below.

Axioms (of identity): $a = a$

Rule of positive conversion (or of simmetry): $\dfrac{a = b}{b = a}$

Rule of transitivity: $\dfrac{a = b \quad b = c}{a = c}$

Rule of positive instantiation: $\dfrac{a = b}{Pa = Pb}$

Rule of assimilation: $\dfrac{PQa = a}{Qa = a}$

Rule of negative conversion: $\dfrac{a \parallel b}{b \parallel a}$

Rule of substitution: $\dfrac{a = b \quad b \parallel c}{a \parallel c}$

Rule of trivialization: $\dfrac{a \parallel a}{\beta}$

We define, in the usual way, when one formula is *deducible* from a set of formulas in our system ($\vdash$). If a set of formulas is such that any formula whatever is deducible from it, the set is called *trivial* or *inconsistent*; otherwise, the set is said to be *consistent*.

We can easily prove the following derived rules:

RD_1: $\dfrac{a = Pb \quad Qa = b}{a = b}$

RD_2: $\dfrac{c = d \quad \alpha}{\beta}$, where β is obtained from α by the replacement of one occurrence of c by one occurrence of d.

We note that we obtain systems equivalent to the one above if: 1) we replace the rule of assimilation by RD_1; 2) we replace the rules of positive conversion, transitivity, positive instanciation and substitution by RD_2.

4. COMPLETENESS OF THE STRONG ALGEBRAIC SYSTEM

Let a be a term and Γ a set of formulas. We put:

$a^\Gamma = \{b$: There exists a functor R such that $\Gamma \vdash b = Ra\} \cup$
$\{\langle b,c\rangle : \Gamma \nvdash b \| c$ and there exists a functor R such that
$\Gamma \vdash b = Ra$ or $\Gamma \vdash c = Ra\}$.

Obviously, $a \in a^\Gamma$.

We shall denote by ψ the set of terms of $\mathcal{L}$.

If P is a functor, we put:

$P^\Gamma = \{\langle u,t\rangle$: Either (i) there exists a term b such that $u = b^\Gamma$ and $t = (Pb)^\Gamma$, or (ii) u is a nonempty subset of $\psi \cup (\psi \times \psi)$ and $u = t$, and there is no term b such that $u = b^\Gamma\}$.

THEOREM 4.1. If $a^\Gamma = b^\Gamma$, then $\Gamma \vdash a = b$.

Proof. Let us suppose that $a^\Gamma = b^\Gamma$. Thus, $a \in b^\Gamma$ and $b \in a^\Gamma$, and therefore there are functors R and S such that $\Gamma \vdash a = Rb$ and $\Gamma \vdash b = Sa$. By the rule of conversion and RD_1, we conclude that $\Gamma \vdash a = b$.

THEOREM 4.2. $\Gamma \vdash a = b$ entails $a^\Gamma = b^\Gamma$.

Proof. Let us suppose that $\Gamma \vdash a = b$. If c is a term such that $c \in a^\Gamma$, then there exists a functor R satisfying the condition $\Gamma \vdash c = \mathrm{R}a$; therefore, by RD_2, $\Gamma \vdash c = \mathrm{R}b$, and $c \in b^\Gamma$. On the other hand, if $\langle c, d \rangle$ is a pair of terms such that $\langle c, d \rangle \in a^\Gamma$, then, $\Gamma \nvdash c \parallel d$ and there is a functor R such that $\Gamma \vdash c = \mathrm{R}a$ or $\Gamma \vdash d = \mathrm{R}a$. By RD_2, $\Gamma \vdash c = \mathrm{R}b$ or $\Gamma \vdash d = \mathrm{R}b$. Thus, $\langle c, d \rangle \in b^\Gamma$ and, consequently, $a^\Gamma \subseteq b^\Gamma$. Analogously, $b^\Gamma \subseteq a^\Gamma$.

THEOREM 4.3. $(\mathrm{P}a)^\Gamma \subseteq a^\Gamma$.

Proof. Immediate, taking into account the definitions of $(\mathrm{P}a)^\Gamma$ and a^Γ.

THEOREM 4.4. P^Γ is a restrictive function on the set $\psi \cup (\psi \times \psi)$.

Proof. We easily show (with the help of Theorem 4.2) that P^Γ is a function, and clearly the domain of P^Γ is the set of all nonempty subsets of $\psi \cup (\psi \times \psi)$. Moreover, if $\langle u, t \rangle \in \mathrm{P}^\Gamma$, then $t \subseteq u$ (by Theorem 4.3) and $u \neq \phi$.

Let Γ be a set of formulas and let $\mathfrak{F}$ be a function which associates with each simple term x of $\mathcal{L}$ the set x^Γ, and with each simple functor P of $\mathcal{L}$ the set P^Γ. The pair $\langle \psi \cup (\psi \times \psi), \mathfrak{F} \rangle$ constitutes an interpretation of $\mathcal{L}$ which we shall denote by I^Γ.

THEOREM 4.5. $I^{\Gamma}(a) = a^{\Gamma}$.

Proof. By induction on the length of a. If a is a simple term, then by definition $I^{\Gamma}(a) = a^{\Gamma}$. Let us suppose that a has the form Pb, where P is a simple functor. By definition, $I^{\Gamma}(Pb) = P^{\Gamma}(I^{\Gamma}(b))$; but $I^{\Gamma}(b) = b^{\Gamma}$, by the induction hypothesis and, by definition, $P^{\Gamma}(b^{\Gamma}) = (Pb)^{\Gamma}$.

THEOREM 4.6. Let Γ be a consistent set of formulas of Γ; $\Gamma \vdash a \parallel b$ if, and only if, $a^{\Gamma} \cap b^{\Gamma} = \varphi$.

Proof. Let us suppose that $\Gamma \nvdash a \parallel b$; it is easily verifiable that $\langle a,b \rangle \in a^{\Gamma}$ and $\langle a,b \rangle \in b^{\Gamma}$.

Now, let c be a term such that $c \in a^{\Gamma}$ and $c \in b^{\Gamma}$. By definition, there are functors R and S such that $\Gamma \vdash c = Ra$ and $\Gamma \vdash c = Sb$; hence, $\Gamma \vdash Ra = Sb$. Therefore, $\Gamma \nvdash Ra \parallel Sb$, since Γ is supposed to be consistent; but the formula $Ra \parallel Sb$ can be derived from $a \parallel b$, and thus we conclude that $\Gamma \nvdash a \parallel b$.

Finally, let c and d be terms such that $\langle c,d \rangle \in a^{\Gamma}$ and $\langle c,d \rangle \in b^{\Gamma}$. By definition:

(i) $\Gamma \nvdash c \parallel d$;

(ii) there is a functor R such that $\Gamma \vdash c = Ra$ or $\Gamma \vdash d = Ra$;

(iii) there is a functor S such that $\Gamma \vdash c = Sb$ or $\Gamma \vdash d = Sb$.

Now, from (ii) and (iii) we conclude that one of the following alternatives obtains:

(I) $\Gamma \vdash Ra = Sb$

(II) $\Gamma \vdash c = Ra$ and $d = Sb$

(III) $\Gamma \vdash d = Ra$ and $c = Sb$.

As we have seen above, (I) is ruled out by the assumed consistency of Γ. From (II) and (i), as well as from (III) and (i), we conclude that $\Gamma \nvdash Ra \parallel Sb$; but this formula can be derived from $a \parallel b$. Hence, $\Gamma \nvdash a \parallel b$.

THEOREM 4.7. If Γ is a consistent set of formulas of $\mathcal{L}$, then $\Gamma \vdash \alpha$ if, and only if, $I^{\Gamma}(\alpha) = 1$.

Proof. By Theorems 4.1, 4.2, 4.5 and 4.6.

THEOREM 4.8. (Completeness). $\Gamma \vdash \alpha$ if, and only if, Γ implies α.

Proof. If $\Gamma \vdash \alpha$, then, by induction on the length of a deduction of α from Γ, we prove that Γ implies α. Let us suppose that Γ implies α; if Γ is inconsistent, then $\Gamma \vdash \alpha$; if Γ is consistent, then I^{Γ} is a model of Γ (Theorem 4.7) and, consequently, $I^{\Gamma}(\alpha) = 1$; by Theorem 4.7, $\Gamma \vdash \alpha$.

THEOREM 4.9. (Completeness, second version). Γ is consistent if, and only if, it has a model.

Proof. By Theorems 4.7 and 4.8.

We remark that our algebraic system may be easily shown to be decidable, by means of an interpretation in the first-order monadic predicate calculus.

5. THE WEAK ALGEBRAIC SYSTEM

The *weak algebraic system* has as its underlying language the language $\mathcal{L}$, and includes, besides the axioms of identity, the rules of positive conversion, transitivity, positive instantiation, negative conversion and substitution.

We define as usual the concept of *deducibility* of a formula α from a set of formulas Γ (and write $\Gamma \overset{*}{\vdash} \alpha$) in the weak algebraic system. We can show that RD_1 is also a derived rule in our new system.

THEOREM 5.1. $\Gamma \overset{*}{\vdash} a = b$ entails $\Gamma^{+} \overset{*}{\vdash} a = b$.

Proof. By induction on the length of a deduction of $a=b$ from Γ.

THEOREM 5.2. If $\Gamma \vdash a \parallel b$, then there exists a negative formula β such that $\Gamma^{+}, \beta \overset{*}{\vdash} a \parallel b$.

Proof. By induction on the length of a deduction of $a \parallel b$ from Γ.

THEOREM 5.3. Let us suppose that S is a simple functor which does not occur in any element of Γ^*. If $\Gamma^* \vdash^{*} a = b$, then either S does not occur in $a = b$ or S occurs both in a and b.

Proof. By induction on the length of the deduction of $a = b$ from Γ^*, under the proviso that S does not occur in any element of Γ^*.

THEOREM 5.4. Let S be a simple functor which does not occur in any element of $\Gamma^* \cup \{a\} \cup \{b\}$. If $\Gamma^* \vdash^{*} RSa = TSb$, then $\Gamma^* \vdash^{*} a = b$.

Proof. By induction on a deduction of $RSa = TSb$ from Γ^*, with the help of Theorem 5.3.

THEOREM 5.5. If $\Gamma^*, c \parallel d \vdash^{*} a \parallel b$, then there exist functors S and T such that either $\Gamma^* \vdash^{*} a = Sc$ and $\Gamma^* \vdash^{*} b = Td$ or $\Gamma^* \vdash^{*} a = Sd$ and $\Gamma^* \vdash^{*} b = Tc$.

Proof. By induction on a deduction of $a \parallel b$ from $\Gamma^* \cup \{c \parallel d\}$.

DEFINITION. Γ will denote a set of formulas of $\mathcal{L}$. A sequence $a_1, a_2, \ldots, a_n$ $(n \geq 1)$ of terms is said to be *Γ-regular* if, for all j $(1 \leq j < n)$, the formula $a_j = a_{j+1}$ is an axiom, or an instance of a member of Γ, or an instance of the converse of a member of Γ. When, for any j and k $(1 \leq j < k \leq n)$, terms a_j and a_k are different, we say that the sequence is *strictly* Γ-*regular*. Obviously, a sequence is Γ-regular if, and only if, it is Γ^*-regular.

THEOREM 5.6. $\Gamma \overset{*}{\vdash} b=c$ if, and only if, there exists a Γ-regular sequence $a_1, a_2, \ldots, a_n$ such that a_1 is b and a_n is c.

Proof. We easily prove by induction on n that, if $a_1, a_2, \ldots, a_n$ is Γ-regular, then $\Gamma \overset{*}{\vdash} a_1 = a_n$. If $\Gamma \overset{*}{\vdash} b=c$, it is also easy to show, by induction on the length of a deduction of $b=c$ from Γ, that there is a sequence $a_1, a_2, \ldots, a_n$ which is Γ-regular and such that a_1 is b and a_n is c.

THEOREM 5.7. Let b and c be two different terms; $\Gamma \overset{*}{\vdash} b=c$ if, and only if, there is a strictly Γ-regular sequence $d_1, d_2, \ldots, d_k$ in which d_1 is b and d_k is c.

Proof. Taking into account Theorem 5.6, it is sufficient to prove that, if $a_1, a_2, \ldots, a_n$ is a Γ-regular sequence such that a_1 and a_n are two different terms, then there exists a strictly Γ-regular sequence $d_1, d_2, \ldots, d_k$ in which a_1 is d_1 and a_n is d_k. This may be proved by induction on n. When $a_1, a_2, \ldots, a_n$ is strictly Γ-regular, there is nothing to be proved. Now, let us suppose that a_1 and a_n are not equal, and that $a_1, a_2, \ldots, a_n$ is not strictly Γ-regular, but only Γ-regular. We consider the least number j such that there exists a number k which satisfies the inequalities $1 \leq j < k$, and a_j is a_k. Three possibilities are then open to us: (i) $j=1$; in this case, evidently $k<n$, $a_k, a_{k+1}, \ldots, a_n$

is Γ-regular, and a_k is a_1; we complete the proof by the application of the inductive hypothesis; (ii) $j \neq 1$ and $k = n$; it is clear that $1 < j < n$, that $a_1, a_2, \ldots, a_j$ is Γ-regular, and that a_j is a_n; we complete the proof as in case (i); (iii) $j \neq 1$ and $k \neq n$; in this case we have that $1 < j < k < n$ and that $a_1, \ldots, a_j, a_{k+1}, \ldots, a_n$ is Γ-regular; we complete the proof as in the preceding cases.

6. THE LANGUAGE $\mathcal{L}_1$ AND ITS INTERPRETATIONS

The vocabulary of $\mathcal{L}_1$ is composed of the simple terms of $\mathcal{L}$ and of the *relation symbols* "A", "E", "I" and "O". The formulas of $\mathcal{L}_1$ are expressions which consist of one relation symbol followed by two distinct simple terms. Unless explicit mention is made to the contrary, the last small Latin letters will be used as variables for (simple) terms, and small Greek letters will denote formulas.

Formulas of the forms Axy and Ixy will be called *positive*, and the others *negative*. If Γ is a set of formulas of $\mathcal{L}_1$, then Γ^+ will denote the set of positive formulas that belong to Γ.

For any formula α of $\mathcal{L}_1$, we define the *contradictory* of α, C(α), as follows: (i) C(Axy) = Oxy; (ii) C(Oxy) = Axy; (iii) C(Exy) = Ixy; (iv) C(Ixy) = Exy.

An interpretation of $\mathcal{L}_1$ is a function from the set of terms of $\mathcal{L}_1$ into a collection of non-empty sets. If $\mathfrak{I}$ is an interpretation of $\mathcal{L}_1$, one defines the value $\mathfrak{I}(\alpha)$ of a formula α by the clauses: (i) $\mathfrak{I}(Axy) = 1$ if, and only if, $\mathfrak{I}(x) \subseteq \mathfrak{I}(y)$; (ii) $\mathfrak{I}(Exy) = 1$ if, and only if, $\mathfrak{I}(x) \cap \mathfrak{I}(y) = \phi$; (iii) $\mathfrak{I}(\alpha) = 1$ if, and only if, $\mathfrak{I}(C(\alpha)) = 0$; (iv) $\mathfrak{I}(\alpha) = 1$ if, and only if, $\mathfrak{I}(\alpha) \neq 0$.

We define in the standard way the concepts of a *model* of a set of formulas and the relation of *implication* between a set of formulas and a formula.

7. THE DIRECT ARISTOTELIAN SYSTEM

The *direct Aristotelian system* (of the syllogistic) has $\mathcal{L}_1$ as its underlying language and possesses the following rules of deduction:

Barbara $\dfrac{Ayz \quad Axy}{Axz}$ Celarent $\dfrac{Eyz \quad Axy}{Exz}$

Darii $\dfrac{Ayz \quad Ixy}{Ixz}$ Ferio $\dfrac{Eyz \quad Ixy}{Oxz}$

Baroco $\dfrac{Azy \quad Oxy}{Oxz}$ Bocardo $\dfrac{Oyz \quad Ayx}{Oxz}$

I-conversion $\dfrac{Ixy}{Iyx}$ E-conversion $\dfrac{Exy}{Eyx}$

AI-subalternation $\dfrac{Axy}{Ixy}$ EO-subalternation $\dfrac{Exy}{Oxy}$

Trivialization $\dfrac{\alpha \quad C(\alpha)}{\beta}$

We define, in the usual way, *deducibility* of a formula α from a set of formulas Γ, in the direct system (we write $\Gamma \overset{*}{\vdash} \alpha$). If the set of formulas Γ (of $\mathcal{L}_1$) is such that $\Gamma \overset{*}{\vdash} \alpha$ for any formula α whatsoever, we say that Γ is *inconsistent**; otherwise, Γ is said to be *consistent**. Obviously, the valid modes of the Aristotelian syllogism are valid deduction rules of the direct system.

It is not difficult to see that, if there exists a deduction of a formula α from the set Γ where the rule of trivialization is employed, then there exists a deduction of α from Γ where this rule is employed only once.

THEOREM 7.1. Γ and α are a set of formulas of $\mathcal{L}_1$ and a positive formula, respectively. If there exists a deduction of α from Γ in the direct system in which the rule of trivialization is not used, then $\Gamma^+ \overset{*}{\vdash} \alpha$.

Proof. By induction on the length of deductions.

COROLLARY. If Γ is a consistent* set of formulas and α is a positive formula, then we have: $\Gamma \overset{*}{\vdash} \alpha$ entails $\Gamma^+ \overset{*}{\vdash} \alpha$.

THEOREM 7.2. Γ is a set of formulas and α is a negative formula. If $\Gamma \overset{*}{\vdash} \alpha$, then there exists a negative formula β such that $\beta \in \Gamma$ and $\Gamma^+, \beta \overset{*}{\vdash} \alpha$

Proof. Suppose that α is negative and $\Gamma \overset{*}{\vdash} \alpha$. It is not difficult to see that there is a deduction of α from Γ in which the rule of trivialization is not employed or is employed just once, precisely in order to justify the one and only occurrence of α in the deduction. The theorem may be trivially proved by induction on the length of this kind of deduction.

COROLLARY 1. $\Gamma^{*} \vdash^{*} \alpha$ entails that α is positive.

COROLLARY 2. Γ^{*} is consistent*.

THEOREM 7.3. Γ is a set of formulas of $\mathcal{L}_1$. $\Gamma^{*} \vdash^{*} Axy$ if, and only if, there exists a sequence of distinct terms $z_1, z_2, \ldots, z_n$ satisfying the following conditions: z_1 is x, z_n is y, and, for any j such that $1 \leq j < n$, $Az_j z_{j+1} \in \Gamma^{*}$.

Proof. Suppose there is a sequence $z_1, z_2, \ldots, z_n$ such as described in the theorem. We show easily, by induction on n, that $\Gamma^{*} \vdash^{*} Az_1 z_n$. Now, suppose that $\Gamma^{*} \vdash^{*} Axy$; first we prove that there is a sequence of terms, not necessarily distinct, $z_1, z_2, \ldots, z_n$ which satisfies the three conditions formulated in the theorem. Finally, we complete the proof by induction on the number of terms of this sequence which occur more than once in it, in a way similar to the one employed in the proof of Theorem 5.7.

If Γ is a set of formulas, we shall represent by Γ^A the set of elements of Γ having the form Axy.

THEOREM 7.4. Γ is a set of formulas of $\mathcal{L}_1$. $\Gamma^* \overset{*}{\vdash} Azw$ if, and only if, $\Gamma^A \overset{*}{\vdash} Azw$.

Proof. Trivial, with the help of Theorem 7.3.

THEOREM 7.5. If Γ is a set of formulas of $\mathcal{L}_1$ and $\Gamma^*, Axy, Ayx \overset{*}{\vdash} Azw$, then $\Gamma^*, Axy \overset{*}{\vdash} Azw$ or $\Gamma^*, Ayx \overset{*}{\vdash} Azw$.

Proof. Let us suppose that $\Gamma^*, Axy, Ayx \overset{*}{\vdash} Azw$; by Theorem 7.3, there exists a sequence of distinct terms $z_1, z_2, \ldots, z_n$, where z_1 is z, z_n is w, and, for every j, $1 \leq j < n$, $Az_j z_{j+1}$ belongs to $\Gamma^* \cup \{Axy, Ayx\}$. We represent by Δ the set $\{\beta$: there is a j such that $1 \leq j < n$ and β is $Az_j z_{j+1}\}$. We prove, by induction on n, that $\Delta \overset{*}{\vdash} Az_1 z_n$, i.e. $\Delta \overset{*}{\vdash} Azw$. But since $z_1, z_2, \ldots, z_n$ are distinct terms, it follows that either $Axy \notin \Delta$ or $Ayx \notin \Delta$. However, $\Delta \subseteq \Gamma^* \cup \{Axy, Ayx\}$; therefore, $\Delta \subseteq \Gamma^* \cup \{Axy\}$ or $\Delta \subseteq \Gamma^* \cup \{Ayx\}$, and so the theorem is proved.

THEOREM 7.6. Γ is a set of formulas of $\mathcal{L}_1$. If $\Gamma^* \overset{*}{\vdash} Ixy$ and $\Gamma^A \overset{*}{\nvdash} Ixy$, then there exist terms z and w such that $Izw \in \Gamma^*$ and $\Gamma^A, Izw \overset{*}{\vdash} Ixy$.

Proof. By induction on the length of a deduction of Ixy from Γ^*.

THEOREM 7.7. Given the set of formulas Γ, if $\Gamma^*, Axy \overset{*}{\vdash} Azw$ and $\Gamma^* \overset{*}{\nvdash} Azw$, then we have: (i) either x is z or $\Gamma^* \overset{*}{\vdash} Azx$; (ii) either y is w or $\Gamma^* \overset{*}{\vdash} Ayw$.

Proof. By induction on the length of a deduction of Azw from $\Gamma^{+} \cup \{Axy\}$.

THEOREM 7.8. Given the set of formulas Γ, if $\Gamma^{+}, Axy \vdash^{*} Izw$ and $\Gamma^{+} \nvdash^{*} Izw$, then we have: either (i) $\Gamma^{+} \vdash^{*} Izx$, if x is not z, and $\Gamma^{+} \vdash^{*} Ayw$, if y is not w, or (ii) $\Gamma^{+} \vdash^{*} Iwx$, if x is not w, and $\Gamma^{+} \vdash^{*} Ayz$, if y is not z.

Proof. We proceed by induction on the length of a deduction of Izw from $\Gamma^{+} \cup \{Axy\}$. Let us assume that $\Gamma^{+} \nvdash^{*} Izw$. If the formula Izw is derived from Azw in the deduction, then we prove the theorem with the help of Theorem 7.7, for the initial assumption implies that $\Gamma \nvdash^{*} Azw$. If Izw is derived from Iwz, the theorem follows trivially from the induction hypothesis. So, let Izw be derived in the deduction by means of the rule DARII. By the induction hypothesis, there is a term t such that $\Gamma^{+}, Axy \vdash^{*} Izt$ and $\Gamma^{+}, Axy \vdash^{*} Atw$. Obviously, our initial assumption ensures that $\Gamma^{+} \nvdash^{*} Izt$ or $\Gamma^{+} \nvdash^{*} Atw$. There remain three cases to be considered:

(a) $\Gamma^{+} \nvdash^{*} Izt$ and $\Gamma^{+} \vdash^{*} Atw$; in this case the theorem is easily derivable by means of the induction hypothesis.

(b) $\Gamma^{+} \vdash^{*} Izt$ and $\Gamma^{+} \nvdash^{*} Atw$; in this case the theorem is easily derivable, with the help of Theorem 7.7.

(c) $\Gamma^{+} \nvdash^{*} Izt$ and $\Gamma^{+} \nvdash^{*} Atw$; by the induction hypothesis, either (i) $\Gamma^{*} \overset{*}{\vdash} Izx$ (if z is not x) and $\Gamma^{*} \overset{*}{\vdash} Ayt$ (if y is not t)

or (ii) $\Gamma^{*} \overset{*}{\vdash} Itx$ (if t is not x) and $\Gamma^{*} \overset{*}{\vdash} Ayz$ (if y is not z) .

Moreover, by Theorem 7.7, we have: (iii) $\Gamma^{*} \overset{*}{\vdash} Ayw$.
Now, (ii) and (iii) imply that $\Gamma^{*} \overset{*}{\vdash} Izw$, which contradicts our initial assumption. So, the alternative (i) obtains; but (i) and (iii) prove the theorem.

THEOREM 7.9. Γ is a set of formulas of $\mathcal{L}_1$. If $\Gamma^{*}, Axy, Ayx \overset{*}{\vdash} Izw$, then either $\Gamma^{*}, Axy \overset{*}{\vdash} Izw$ or $\Gamma^{*}, Ayx \overset{*}{\vdash} Izw$.

Proof. By induction on the length of a deduction of Izw from Γ^{*}, Axy, Ayx. If $Izw \in \Gamma^{*}$, then the theorem is trivially provable. If Izw is derived in the deduction by conversion or subalternation, then the theorem is easily provable by means of the induction hypothesis or Theorem 7.5, respectively. Let us assume that Izw is derived in the deduction by DARII. There is a term t such that $\Gamma^{*}, Axy, Ayx \overset{*}{\vdash} Atw$ and $\Gamma^{*}; Axy, Ayx \overset{*}{\vdash} Izt$. In view of the induction hypothesis and Theorem 7.5, there are four cases to be considered

(a) $\Gamma^{*}, Axy \overset{*}{\vdash} Atw$ and $\Gamma^{*}, Axy \overset{*}{\vdash} Izt$; obviously, $\Gamma^{*}, Axy \overset{*}{\vdash} Izw$.

(b) $\Gamma^{*}, Ayx \overset{*}{\vdash} Atw$ and $\Gamma^{*}, Ayx \overset{*}{\vdash} Izt$; obviously, $\Gamma^{*}, Ayx \overset{*}{\vdash} Izw$.

(c) $\Gamma^{*}, Ayx \overset{*}{\vdash} Atw$ and $\Gamma^{*}, Axy \overset{*}{\vdash} Izt$; if $\Gamma^{*} \overset{*}{\vdash} Izt$ or

$\Gamma^* \overset{*}{\vdash} Atw$, then the theorem can be proved as it was in (a) and (b) above; so, let $\Gamma^* \overset{*}{\nvdash} Izt$ and $\Gamma^* \overset{*}{\nvdash} Atw$. By Theorem 7.5, we have: (i) $\Gamma^* \overset{*}{\vdash} Axw$ (if x is not w) and (ii) $\Gamma^* \overset{*}{\vdash} Aty$ (if t is not y). Now let $\Gamma^* \overset{*}{\vdash} Izx$ or x be z; by (i), $\Gamma^* \overset{*}{\vdash} Izw$; but if x is not z and $\Gamma^* \overset{*}{\nvdash} Izx$, then, by Theorem 7.8, $\Gamma^* \overset{*}{\vdash} Itx$ (if x is not t) and $\Gamma^* \overset{*}{\vdash} Ayz$ (if y is not z). Together with (i) and (ii), this implies that $\Gamma^* \overset{*}{\vdash} Izw$.

(d) $\Gamma^*, Axy \overset{*}{\vdash} Atw$ and $\Gamma^*, Ayx \overset{*}{\vdash} Izt$; in this case, the proof is analogous to that of case (c).

8. THE INDIRECT ARISTOTELIAN SYSTEM

The indirect Aristotelian system, essentially that of Corcoran (cf. Corcoran 1972), is formulated in $\mathcal{L}_1$. A *proof* in it is a finite sequence of ordered pairs $\langle \alpha, \Gamma \rangle$, where α is a formula of $\mathcal{L}_1$ and Γ is a finite set of formulas, such that: (i) $\alpha \in \Gamma$; or (ii) there exist two previous pairs in the sequence, $\langle \beta, \psi \rangle$ and $\langle \gamma, \Delta \rangle$, where $\Gamma = \psi \cup \Delta$ and α is obtained from β and γ by *Barbara* or *Darii*; or (iii) there exists a formula β such that α is obtained from β by the rule of I-conversion or by the rule of AI-subalternation, and the pair $\langle \beta, \Gamma \rangle$ occurs previously in the sequence; or (iv) there are two preceding pairs in the sequence, $\langle \beta, \psi \rangle$ and $\langle \gamma, \psi \rangle$, where $\psi = \Gamma \cup \{C(\alpha)\}$ and $\beta = C(\gamma)$.

If $\langle \alpha, \Gamma \rangle$ is the last pair of a proof in the indirect system, this proof is called a proof of $\langle \alpha, \Gamma \rangle$ in the system. If there exists a proof of $\langle \alpha, \Gamma \rangle$ and Δ is a set of formulas of $\mathcal{L}_1$ such that $\Gamma \subseteq \Delta$, then we say that α is a *consequence* of (or is *deducible*) from Δ, and we write: $\Delta \vdash \alpha$. A set of formulas ψ is called inconsistent if $\psi \vdash \gamma$ for any formula γ whatsoever; otherwise, ψ is said to be consistent.

Let Γ and Δ be finite sets of formulas of $\mathcal{L}_1$. We easily prove that if there exists a proof of $\langle \alpha, \Gamma \rangle$ in the indirect system, then there is a proof of $\langle \alpha, \Gamma \cup \Delta \rangle$. Therefore, if Δ is finite, $\Gamma \subseteq \Delta$ and there exists a proof of $\langle \alpha, \Gamma \rangle$, then there is also a proof of $\langle \alpha, \Delta \rangle$. If Δ is finite, $\Delta \vdash \alpha$ if, and only if, there is a proof of $\langle \alpha, \Delta \rangle$.

THEOREM 8.1. $\Gamma \cup \{\alpha\}$ is a set of formulas of $\mathcal{L}_1$. If $\Gamma \vdash^{*} \alpha$, then $\Gamma \vdash \alpha$.

Proof. By induction on a proof of α from Γ.

Therefore, the valid modes of the Aristotelian syllogistic constitute valid rules of the indirect system.

THEOREM 8.2. If $\Gamma, \alpha \vdash \beta$, then $\Gamma, C(\beta) \vdash C(\alpha)$.

Proof. Immediate.

9. COMPLETENESS OF THE INDIRECT SYSTEM

Let Γ be a set of formulas of $\mathcal{L}_1$. Given the term x, we put:

$$x^{(\Gamma)} = \{x\} \cup \{y: \Gamma \vdash Ayx\},$$

and

$$x^{\Gamma} = x^{(\Gamma)} \cup \{\langle z,w\rangle: \Gamma \vdash Izw \text{ and } (z \in x^{(\Gamma)} \text{ or } w \in x^{(\Gamma)})\}.$$

THEOREM 9.1. $\Gamma \vdash Axy$ if, and only if, $x^{\Gamma} \subseteq y^{\Gamma}$, for any two distinct terms x and y.

Proof. (i) $\Gamma \vdash Axy$ and z is any element of $x^{(\Gamma)}$; if z is x or y, then obviously $z \in y^{(\Gamma)}$; if not, $\Gamma \vdash Azx$ and, by *Barbara*, $\Gamma \vdash Azy$, and $z \in y^{(\Gamma)}$; hence $x^{(\Gamma)} \subseteq y^{(\Gamma)}$.
(ii) $\Gamma \vdash Axy$ and z and w are such that $\langle z,w\rangle \in x^{\Gamma}$; then, by (i) above, we clearly have that $\langle z,w\rangle \in y^{\Gamma}$; so $x^{\Gamma} \subseteq y^{\Gamma}$.
(iii) If $x^{\Gamma} \subseteq y^{\Gamma}$, then, since $x \in x^{\Gamma}$, we have that $x \in y^{\Gamma}$ and $\Gamma \vdash Axy$, if x is not y.

THEOREM 9.2. $\Gamma \vdash Ixy$ if, and only if, $x^{\Gamma} \cap y^{\Gamma} \neq \phi$, where x and y are distinct terms.

Proof. (i) If $\Gamma \vdash Ixy$, then, by definition, $\langle x,y\rangle \in x^{\Gamma}$ and $\langle x,y\rangle \in y^{\Gamma}$. (ii) Let $x^{(\Gamma)} \cap y^{(\Gamma)} \neq \phi$; there exists a term z such that $z \in x^{(\Gamma)}$ and $z \in y^{(\Gamma)}$; if z is x, then $\Gamma \vdash Axy$ and, hence, $\Gamma \vdash Ixy$ (subalternation); if z is y, then $\Gamma \vdash Ayx$ and, therefore, $\Gamma \vdash Ixy$ (subalternation and conversion); if z is neither x nor y, then $\Gamma \vdash Azx$ and $\Gamma \vdash Azy$; so, $\Gamma \vdash Ixy$ (*Darapti*). (iii) Suppose there are terms z and w such that $\langle z,w\rangle \in x^{\Gamma}$ and $\langle z,w\rangle \in y^{\Gamma}$. By definition, $\Gamma \vdash Izw$, and: (a) $z \in x^{(r)}$ and $z \in y^{(r)}$; we have that $\Gamma \vdash Ixy$, by (ii) above; or (b) $w \in x^{(r)}$ and $w \in y^{(r)}$; in this case, $\Gamma \vdash Ixy$ by (ii) above; or (c) $z \in x^{(r)}$ and $w \in y^{(r)}$; if z is x and w is y, we get $\Gamma \vdash Ixy$; if z is x and w is not y, we get $\Gamma \vdash Ixw$ and $\Gamma \vdash Awy$, and therefore $\Gamma \vdash Ixy$ (*Darii*); when z is not x and w is y, we have $\Gamma \vdash Azx$ and $\Gamma \vdash Izy$, and $\Gamma \vdash Ixy$ (*Disamis*); when z is not x and w is not y, then $\Gamma \vdash Azx$ and $\Gamma \vdash Awy$. Thus, since $\Gamma \vdash Izw$, we have that $\Gamma \vdash Ixy$ (*Darii* and *Disamis*); or (d) $z \in y^{(r)}$ and $w \in x^{(r)}$; similar to case (c). To conclude, then, $\Gamma \vdash Ixy$ if, and only if, $x^{\Gamma} \cap y^{\Gamma} \neq \phi$.

THEOREM 9.3. Γ is a consistent set of formulas of $\mathcal{L}_1$. We have: 1) If $\Gamma \vdash Exy$, then $x^{\Gamma} \cap y^{\Gamma} = \phi$; 2) If $\Gamma \vdash Oxy$, then $x^{\Gamma} \not\subseteq y^{\Gamma}$.

Proof. Immediate.

Let Γ be a consistent set of formulas of $\mathcal{L}_1$. We shall denote by $\mathfrak{I}^{\Gamma}$ the function which associates with each term x of $\mathcal{L}_1$ the set x^{Γ}. This function is an interpretation of $\mathcal{L}_1$.

THEOREM 9.4. $\mathfrak{I}^{\Gamma}$ is a model of Γ whenever Γ is consistent.

Proof. By Theorems 9.1 to 9.3.

THEOREM 9.5. If $\Gamma \vdash \alpha$, then Γ implies α.

Proof. By induction, in the usual manner.

THEOREM 9.6. Γ is consistent if, and only if, Γ has a model.

Proof. Theorems 9.4 and 9.5.

THEOREM 9.7. $\Gamma \cup \{C(\alpha)\}$ is inconsistent if, and only if $\Gamma \vdash \alpha$.

THEOREM 9.8. If Γ implies α, then $\Gamma \vdash \alpha$.

10. THE WEAK ALGEBRAIC SYSTEM AND THE ARISTOTELIAN SYSTEMS

We suppose as given an enumeration of the formulas of $\mathcal{L}_1$. For any formula α of $\mathcal{L}_1$ we define the *translation of* α *in* $\mathcal{L}$, in symbols $tr(\alpha)$, as follows (j is the number of α in the given enumeration): (i) if α is Axy, then $tr(\alpha)$ is $x = Q_j y$; (ii) if α is Ixy, then $tr(\alpha)$ is $P_j x = Q_j y$; (iii) if α is Exy, then $tr(\alpha)$ is $x \parallel y$; (iv) if α is Oxy, then $tr(\alpha)$ is $P_j x \parallel y$.

If Γ is a set of formulas of $\mathcal{L}_1$, we denote by $tr(\Gamma)$ the set of formulas β such that, for some element α of Γ, β is $tr(\alpha)$.

A formula α of $\mathcal{L}_1$ is said to be a *version* of a formula β of $\mathcal{L}$ if: (i) α is Axy and there is a non-empty functor R of $\mathcal{L}$ such that β is $x = Ry$; or (ii) α is Ixy and there exist two non-empty functors R and S of $\mathcal{L}$ such that β is $Rx = Sy$; or (iii) α is Exy and β is $x \parallel y$; or (iv) α is

Oxy and there is a non-empty functor R of $\mathcal{L}$ such that β is $Rx \parallel y$.

We remark that if Γ is a set of formulas of $\mathcal{L}_1$, then $tr(\Gamma)$ is a set of formulas of $\mathcal{L}$ satisfying the condition that each element α of $tr(\Gamma)$ is not an instance of any other formula of $\mathcal{L}$. Therefore, if $a_1, a_2, \dots, a_n$ is a $tr(\Gamma)$-regular sequence and $1 \leq k < n$, then the origin of $a_k = a_{k+1}$ is an element of $tr(\Gamma)$ or is the converse of an element of $tr(\Gamma)$ or is an axiom (when a_k is a_{k+1}).

THEOREM 10.1. Let Γ be a set of formulas of $\mathcal{L}_1$, $a_1, a_2, \dots, a_n$ a strictly $tr(\Gamma)$-regular sequence with $n > 2$, and k a number such that $1 \leq k < (n-1)$. If the origin of $a_k = a_{k+1}$ is $b = c$, then the origin of $a_{k+1} = a_{k+2}$ is not $c = b$.

Proof. Given the conditions expressed by the hypothesis of the theorem, let us suppose that the origin of $a_k = a_{k+1}$ is $b = c$ and that the origin of $a_{k+1} = a_{k+2}$ is $c = b$. $a_k = a_{k+1}$ is of the form $Rb = Rc$ and $a_{k+1} = a_{k+2}$ has the form $Sc = Sb$. But in this case, R is S and, consequently, a_k is a_{k+2}, which is absurd, since the sequence is strictly regular.

THEOREM 10.2. Γ denotes a set of formulas of $\mathcal{L}_1$, $a_1, a_2, \dots, a_n$ is a strictly $tr(\Gamma)$-regular sequence, and x_j is the simple term that occurs in a_j $(1 \leq j \leq n)$. If the origin of $a_1 = a_2$ has the form $x_1 = Qx_2$, then for all k, $1 \leq k < n$, the origin of $a_k = a_{k+1}$ has the form $x_k = Rx_{k+1}$, for some simple functor R.

Proof. Given the conditions expressed by the hypothesis of the theorem, let us suppose that the origin of $a_1 = a_2$ has the form $x_1 = Qx_2$. Given that a_1 is not a_2, we conclude that Q is a simple functor, by the definition of $tr(\Gamma)$; if $n = 2$, we have nothing to prove. If $n > 2$, we prove the theorem by induction on n; since $a_2, \ldots, a_n$ is also a strictly $tr(\Gamma)$-regular sequence, it is enough to prove that $a_2 = a_3$ has as its origin a formula of $\mathcal{L}$ of the form $x_2 = Rx_3$, for some simple functor R. By the definition of a strictly regular sequence, the origin of $a_2 = a_3$ is not an axiom, hence it is an element of $tr(\Gamma)$ or the converse of such an element. Thus, it is either of the form $x_2 = Rx_3$, for some simple functor R, or is of the form $Sx_2 = b$, for some simple functor S. If the origin of $a_2 = a_3$ is of the form $Sx_2 = b$, where S is a simple functor, since $a_1 = a_2$ is, by hypothesis, an instance of $x_1 = Qx_2$, where Q is a simple functor, we conclude that Qx_2 is Sx_2 and, therefore, that Q is S. However, by the definition of $tr(\Gamma)$, a simple functor cannot occur in more than one element of $tr(\Gamma)$. We arrive at the conclusion, then, that $Sx_2 = b$ is the converse of $x_1 = Qx_2$, which is absurd, by Theorem 10.1.

THEOREM 10.3. Γ is a set of formulas of $\mathcal{L}_1$, $a_1, a_2, \ldots, a_n$ is a strictly $tr(\Gamma)$-regular sequence $(n > 2)$ and x_j is the simple term that occurs in a_j $(1 \leq j \leq n)$. If the origin of $a_1 = a_2$ has the form $Px_1 = Qx_2$, where P and Q are two simple functors, then the origin of $a_2 = a_3$ has the form $x_2 = Rx_3$ for some simple functor R.

Proof. Analogous to the last part of the proof of Theorem 10.2.

THEOREM 10.4. Given the conditions of Theorem 10.2: if the origin of $a_1 = a_2$ has the form $x_1 = Qx_2$, then x_1 is x_n or $\Gamma \overset{*}{\vdash} Ax_1x_n$.

Proof. If the hypothesis of the theorem is satisfied, let us suppose that the origin of $a_1 = a_2$ has the form $x_1 = Qx_2$. By the definition of $tr(\Gamma)$, $Ax_1x_2 \in \Gamma$ and, consequently, $\Gamma \overset{*}{\vdash} Ax_1x_2$. When $n = 2$, the theorem is proved. When $n > 2$, we prove the theorem by induction on n. The induction hypothesis and Theorem 10.2 show that x_2 is x_n or $\Gamma \overset{*}{\vdash} Ax_2x_n$ (because $a_2, \dots, a_n$ is a strictly $tr(\Gamma)$-regular sequence of length less than n). If x_2 is x_n the theorem is proved; otherwise, we conclude that $\Gamma \overset{*}{\vdash} Ax_1x_n$ (by *Barbara*) or x_1 is x_n.

THEOREM 10.5. Given the hypothesis of Theorem 10.2: if the origin of $a_1 = a_2$ is of the form $Px_1 = Qx_2$, where P and Q are simple functors, we have: x_1 is x_n or $\Gamma \overset{*}{\vdash} Ix_1x_n$.

Proof. Analogous to the proof of Theorem 10.4, with the help of Theorem 10.3 and 10.4.

THEOREM 10.6. Let Γ be a set of formulas of $\mathcal{L}_1$, x a simple term, and b a term of $\mathcal{L}$. If $tr(\Gamma) \overset{*}{\vdash} x = b$, then b is x or b is not a simple term.

Proof. With the help of Theorems 5.7 and 10.2.

THEOREM 10.7. Given the conditions of Theorem 10.2: if the origin of $a_1 = a_2$ is of the form $Px_1 = x_2$, then x_1 is x_n or $\Gamma \overset{*}{\vdash} Ix_1x_n$.

Proof. By the definition of $tr(\Gamma)$, $Ax_2x_1 \in \Gamma$, hence $\Gamma \overset{*}{\vdash} Ix_1x_2$. When $n = 2$, the theorem is proved. When $n > 2$, we reason by induction on n. We have three cases: (a) the origin of $a_2 = a_3$ has the form $x_2 = Qx_3$ and, by Theorem 10.4, x_2 is x_n or $\Gamma \overset{*}{\vdash} Ax_2x_n$. Thus, since $\Gamma \overset{*}{\vdash} Ix_1x_2$, we conclude that x_1 is x_n or $\Gamma \overset{*}{\vdash} Ix_1x_n$; (b) the origin of $a_2 = a_3$ has the form $Rx_2 = Qx_3$, where R and Q are simple functors. In this case, by Theorem 10.5, x_2 is x_n or $\Gamma \overset{*}{\vdash} Ix_2x_n$. Since $\Gamma \overset{*}{\vdash} Ax_2x_1$, it follows that x_1 is x_n or $\Gamma \overset{*}{\vdash} Ix_1x_n$; (c) the origin of $a_2 = a_3$ has the form $Rx_2 = x_3$; by the induction hypothesis, x_2 is x_n or $\Gamma \overset{*}{\vdash} Ix_2x_n$. Since $\Gamma \overset{*}{\vdash} Ax_2x_1$, we see that x_1 is x_n or $\Gamma \overset{*}{\vdash} Ix_1x_n$.

THEOREM 10.8. $\Gamma \cup \{\beta\}$ is a set of formulas of $\mathcal{L}_1$. $\Gamma^* \overset{*}{\vdash} \beta$ if, and only if, there exists a formula α of $\mathcal{L}$ such that β is a version of α, α is positive, and $tr(\Gamma) \overset{*}{\vdash} \alpha$.

Proof. (i) One easily proves that if $\Gamma^* \overset{*}{\vdash} \beta$, then there exists a formula α of $\mathcal{L}$ such that β is a version of α, α is positive (Theorem 7.2, Corollary 1) and $tr(\Gamma) \overset{*}{\vdash} \alpha$;
(ii) if there exists a positive formula α of $\mathcal{L}$, of which β is a version, $tr(\Gamma) \overset{*}{\vdash} \alpha$ and α has the form $b = c$, then there exists a strictly $tr(\Gamma)$-regular sequence $a_1, a_2, \dots, a_n$ such that a_1 is b and a_n is c, by Theorem 5.7 (the fact that β is a version of $b = c$ assures us that b and c are

distinct terms). We have three possible cases: (a) the origin $a_1 = a_2$ has the form $x_1 = Qx_2$, where Q is a simple functor. Since x_1 is not x_n, because $a_1 = a_n$ has the version β in $\mathcal{L}_1$, Theorem 10.4 implies that $\Gamma \overset{*}{\vdash} Ax_1x_n$ and, therefore, that $\Gamma \overset{*}{\vdash} Ix_1x_n$; but β is Ax_1x_n or Ix_1x_n, because it is a version of $a_1 = a_n$; (b) the origin of $a_1 = a_2$ has the form $Px_1 = Qx_2$, with P and Q simple functors; x_1 is not x_n and, by Theorem 10.5, we have $\Gamma \overset{*}{\vdash} Ix_1x_n$; but β is Ix_1x_n; (c) the origin of $a_1 = a_2$ has the form $Px_1 = x_2$, where P is a simple functor; as in case (b), Theorem 10.7 entails that $\Gamma \overset{*}{\vdash} \beta$.

THEOREM 10.9. $\Gamma \cup \{\beta\}$ is a set of formulas of $\mathcal{L}_1$. If there exists a proof of β from Γ in the direct system in which the rule of trivialization is not used, then $tr(\Gamma) \overset{*}{\vdash} \alpha$, for some α of $\mathcal{L}$ such that β is a version of α.

Proof. By induction on a deduction of β from Γ.

THEOREM 10.10. Let Γ be a set of formulas of $\mathcal{L}_1$. If Γ is inconsistent*, then there exists a term a of $\mathcal{L}$ such that $tr(\Gamma) \overset{*}{\vdash} a \parallel a$.

Proof. If Γ is inconsistent*, it is easy to see that there exists a formula α of $\mathcal{L}_1$ such that $\Gamma \overset{*}{\vdash} \alpha$, $\Gamma \overset{*}{\vdash} C(\alpha)$, and both α and $C(\alpha)$ can be deduced from Γ in the direct system, without any application of the rule of trivialization. We may suppose that α is positive. By Theorem 10.9, $tr(\Gamma) \overset{*}{\vdash} \beta$ and $tr(\Gamma) \overset{*}{\vdash} \gamma$, where α is a version of β and $C(\alpha)$ a version of γ. Two alternatives are possible: (a) α

is Axy ; β then has the form $x = Ry$ and γ has the form $Sx \parallel y$, hence $tr(\Gamma) \overset{*}{\vdash} Sx \parallel Sx$; (ii) α is Ixy ; β has the form $Tx = Ry$ and γ the form $x \parallel y$, thus $tr(\Gamma) Tx \parallel Tx$.

THEOREM 10.11. Γ is a set of formulas of $\mathcal{L}_1$ and β is a negative formula of this language. If there exists a formula α of $\mathcal{L}$ such that β is a version of α and $tr(\Gamma) \overset{*}{\vdash} \alpha$, then $\Gamma \overset{*}{\vdash} \beta$.

Proof. Suppose there is a negative formula α of $tr(\Gamma)$ such that β is a version of α and $tr(\Gamma) \overset{*}{\vdash} \alpha$. The formula β has either the form Exy or the form Oxy. By Theorem 5.2, there is a negative element γ of $tr(\Gamma)$ such that $tr(\Gamma)^{+}, \gamma \overset{*}{\vdash} \alpha$. There are two cases to be considered.

(a) γ is tr(Ezw), thus γ is $z \parallel w$ and we have

(i) $tr(\Gamma)^{+}, z \parallel w \overset{*}{\vdash} \alpha$;

(ii) $Ezw \in \Gamma$.

Let β be Exy; then α will be $x \parallel y$; (i), Theorems 5.5, 10.6 and 10.8 ensure that either $\Gamma \overset{*}{\vdash}$ Axz (if x is not z) and $\Gamma \overset{*}{\vdash}$ Ayw (if y is not w) or $\Gamma \overset{*}{\vdash}$ Axw (if x is not w) and $\Gamma \overset{*}{\vdash}$ Ayz (if y is not z). Together with (ii), this leads easily to the conclusion that $\Gamma \overset{*}{\vdash}$ Exy. Now let β be Oxy. This means that α is $Sx \parallel y$, for some non-empty functor S. From (i), Theorems 5.5, 10.6 and 10.8 ensure that either $\Gamma \overset{*}{\vdash}$ Ixz (if x is not z) and $\Gamma \overset{*}{\vdash}$ Ayw (if y is not w) or $\Gamma \overset{*}{\vdash}$ Ixw (if x is not w) and $\Gamma \overset{*}{\vdash}$ Ayz (if y is not z). Combined with (ii), this easily leads to the conclusion that $\Gamma \overset{*}{\vdash}$ Oxy .

(b) γ is tr(Ozw). Thus it is $Pz \parallel w$, for some simple functor P which occurs in no element of $tr(\Gamma)^*$ and we have:

(i) $tr(\Gamma)^*, Pz \parallel w \overset{*}{\vdash} \alpha$;

(ii) $Ozw \in \Gamma$.

We can show that β cannot be Exy. For suppose it were, then the formula α would be $x \parallel y$. By (i) and Theorem 5.5, we would derive that, for some functor R, $tr(\Gamma)^* \overset{*}{\vdash} x = RPz$ or $tr(\Gamma)^* \overset{*}{\vdash} y = RPz$; but this conclusion is made impossible by Theorem 5.3. Hence, β is Oxy and α is $Sx \parallel y$, for some non-empty functor S. As it is not the case, for all functors R, that $tr(\Gamma)^* \overset{*}{\vdash} y = RPz$ (Theorem 5.3), we obtain from (i) and Theorem 5.5 that:

(iii) $tr(\Gamma)^* \overset{*}{\vdash} Sx = RPz,\ y = Qw$, for some functor R and some non-empty (Theorem 10.6) functor Q.

This and Theorems 5.4 and 10.6 imply that:

(iv) $tr(\Gamma)^* \overset{*}{\vdash} z = Tx$, for some non-empty functor T.

From (iii), (iv) and Theorem 10.8, we derive that $\Gamma \overset{*}{\vdash} Azx$ and $\Gamma \overset{*}{\vdash} Ayw$. Together with (ii), this leads to the conclusion that $\Gamma \overset{*}{\vdash} Oxy$.

THEOREM 10.12. Γ is a set of formulas of $\mathcal{L}$ and a,b and c are terms of $\mathcal{L}$. If, $\Gamma^*, b \parallel c \overset{*}{\vdash} a \parallel a$, then there exist non-empty functors R and S such that $\Gamma^* \overset{*}{\vdash} Rb = Sc$.

Proof With the help of Theorem 5.5.

THEOREM 10.13 Let Γ be a set of formulas of $\mathcal{L}_1$, x and y two distinct simple terms, P a simple functor which does not occur in any element of $tr(\Gamma)^*$, and a a term of $\mathcal{L}$. If $tr(\Gamma)^*, Px \| y \stackrel{*}{\vdash} a \| a$, then there exists a non-empty functor R such that $tr(\Gamma)^* \stackrel{*}{\vdash} x = Ry$.

Proof. If $tr(\Gamma)^*, Px \| y \stackrel{*}{\vdash} a \| a$, then by Theorem 10.12 there exist non-empty functors S and T such that $tr(\Gamma)^* \stackrel{*}{\vdash} SPx = Ty$. By Theorem 5.3, Ty can be written as QPUy, where Q is a functor and U a functor in which P does not occur. By Theorem 5.4, $tr(\Gamma)^* \stackrel{*}{\vdash} x = Uy$ and, by Theorem 10.6, U is not empty.

THEOREM 10.14. The set of formulas Γ of $\mathcal{L}_1$ is inconsistent* if, and only if, there exists a term a of $\mathcal{L}$ such that $tr(\Gamma) \stackrel{*}{\vdash} a \| a$.

Proof. By Theorem 10.10, it suffices to show that Γ is inconsistent* if there is an a such that $tr(\Gamma) \stackrel{*}{\vdash} a \| a$. If there exists such a term, then, by Theorem 5.2, there exists a negative formula β in $tr(\Gamma)$ satisfying the condition: $tr(\Gamma)^*, \beta \stackrel{*}{\vdash} a \| a$. Hence, we have that either β is of the form $x \| y$, where x and y are distinct simple terms, or β is of the form $Px \| y$, where x and y are two distinct simple terms, and P a simple functor that does not occur in any element of $tr(\Gamma)^*$. In the first hypothesis, $\Gamma \stackrel{*}{\vdash} Exy$ and, by Theorems 10.8 and 10.12, $\Gamma \stackrel{*}{\vdash} Ixy$; in the second, $\Gamma \stackrel{*}{\vdash} Oxy$ and, by Theorems 10.8 and 10.13, $\Gamma \stackrel{*}{\vdash} Axy$. In both hypotheses, Γ is inconsistent*.

THEOREM 10.15. Γ is a consistent* set of formulas of $\mathcal{L}_1$, x and y are simple terms, and a is a term of $\mathcal{L}$. If $tr(\Gamma), tr(Axy) \overset{*}{\vdash} a \| a$, then $tr(\Gamma) \overset{*}{\vdash} \beta$, for some formula β such that Oxy is the version of β.

Proof. Suppose that $tr(\Gamma), tr(Axy) \overset{*}{\vdash} a \| a$. Since Γ is consistent*, by Theorem 10.14 $Axy \notin \Gamma$. By Theorem 5.2, there exists a negative element γ of $tr(\Gamma)$ such that $tr(\Gamma^* \cup \{Axy\}), \gamma \overset{*}{\vdash} a \| a$. Two cases are possible: γ is $z \| w$ or γ is $Pz \| w$, for some simple functor P which does not occur in any element of the set $tr(\Gamma^* \cup \{Axy\})$ (obviously, $tr(\Gamma^* \cup \{Axy\})$ is identical to $tr(\Gamma \cup \{Axy\})^*$).

By Theorem 10.12, either there are functors R and S such that $tr(\Gamma \cup \{Axy\})^* \overset{*}{\vdash} Rz = Sw$ (if γ is $z \| w$) or $tr(\Gamma \cup \{Axy\})^* \overset{*}{\vdash} RPz = Sw$ (if γ is $Pz \| w$). Hence, by Theorem 5.7, there exists a strictly $tr(\Gamma \cup \{Axy\})^*$-regular sequence $b_1, b_2, \dots, b_n$ such that b_1 is Rz (when γ is $z \| w$) or RPz (when γ is $Pz \| w$), and b_n is Sw. Since $tr(\Gamma)^* \overset{*}{\nvdash} b_1 = b_n$, because $tr(\Gamma) \overset{*}{\vdash} Rz \| Sw$ (if γ is $z \| w$) or $tr(\Gamma) \overset{*}{\vdash} RPz \| Sw$ (if γ is $Pz \| w$) and Γ is consistent, we conclude that there exists a j ($1 \leqslant j < n$) for which $b_1 = b_{j+1}$ has as origin tr(Axy) or the converse of tr(Axy).

Let tr(Axy) be of the form $x = Qy$, where Q is a simple functor which does not occur in any element of $tr(\Gamma)$. There exists the least number h such that $1 \leqslant h < n$ and the origin of $b_h = b_{h+1}$ is $x = Qy$ or $Qy = x$, and there exists the greatest number k such that $1 \leqslant k < n$ and the origin of $b_k = b_{k+1}$ is $x = Qy$ or $Qy = x$. Clearly, the sequences

$b_1, b_2, \ldots, b_h$ (b_1, b_1 if $h = 1$) and $b_{k+1}, b_{k+2}, \ldots, b_n$ (b_n, b_n if $k+1 = n$) are $tr(\Gamma)^*$-regular. Therefore, by Theorem 5.6:

(i) $tr(\Gamma)^* \overset{*}{\vdash} b_1 = b_h$ and $tr(\Gamma)^* \overset{*}{\vdash} b_{k+1} = b_n$.

Let the origin of $b_h = b_{h+1}$ be $Qy = x$. We show that the origin of $b_k = b_{k+1}$ cannot be $x = Qy$. Suppose it were, then by (i), we would have:

(ii) $tr(\Gamma)^* \overset{*}{\vdash} UQy = Sw$, for some functor U;

(iii) if γ is $z \parallel w$, then $tr(\Gamma)^* \overset{*}{\vdash} Rz = VQy$, for some functor V;

(iv) if γ is $Pz \parallel w$, then $tr(\Gamma)^* \overset{*}{\vdash} RPz = VQy$, for some functor V.

Together with Theorems 5.3 and 5.4, (ii) implies that $tr(\Gamma)^* \overset{*}{\vdash} y = Mw$, for some functor M; (iii) implies that, if γ is $z \parallel w$, then $tr(\Gamma)^* \overset{*}{\vdash} y = Tz$, for some functor T; (iv) implies that, if γ is $Pz \parallel w$, then $tr(\Gamma)^* \overset{*}{\vdash} y = z$. From this, it is not difficult to derive the result that $tr(\Gamma) \overset{*}{\vdash} y \parallel y$ and, therefore, by means of Theorem 10.14, that Γ would be inconsistent*.

On the other hand, if the origin of $b_h = b_{h+1}$ is $x = Qy$, then the origin of $b_k = b_{k+1}$ can not be $Qy = x$, in view of Theorem 10.2. There remain, then, two alternatives:

(a) The origin of $b_k = b_{k+1}$ and $b_h = b_{h+1}$ is $x = Qy$: in this case, $tr(\Gamma)^* \overset{*}{\vdash} TQy = Sw$, for some functor T, and, by Theorem 5.3 and 5.4, $tr(\Gamma)^* \overset{*}{\vdash} y = Vw$, for some functor V. Furthermore, either $tr(\Gamma)^* \overset{*}{\vdash} Rz = Ux$ (if γ is $z \parallel w$) or $tr(\Gamma)^* \overset{*}{\vdash} RPz = Ux$ (if γ is $Pz \parallel w$), for some functor U. Since $tr(\Gamma) \overset{*}{\vdash} Rz \parallel Vw$ (when γ is $z \parallel w$) or $tr(\Gamma) \overset{*}{\vdash} RPz \parallel Vw$ (when

γ is $Pz \parallel w$), it follows that $tr(\Gamma) \vdash^{*} Ux \parallel y$ and, therefore, that $tr(\Gamma) \vdash^{*} PUx \parallel y$; but the version of $PUx \parallel y$ is Oxy.

(b) The origin of $b_h = b_{h+1}$ and $b_k = b_{k+1}$ is $Qy = x$; in this case, $tr(\Gamma)^{+} \vdash^{*} Ux = Sw$, for some functor U. However, γ cannot be $Pz \parallel w$, because otherwise we would have $tr(\Gamma)^{+} \vdash^{*} RPz = TQy$, for some functor T, and this is absurd by Theorems 5.3 and 5.4, taking into account that P and Q are distinct simple functors which do not occur in any element of $tr(\Gamma)^{+}$. We have, then, that γ is $z \parallel w$ and, consequently, that $tr(\Gamma)^{+} \vdash^{*} Rz = TQy$, for some functor T. By Theorems 5.3 and 5.4, $tr(\Gamma)^{+} \vdash^{*} y = Vz$, for some functor V. Since $tr(\Gamma) \vdash^{*} Vz \parallel Sw$, it follows that $tr(\Gamma) \vdash^{*} Ux \parallel y$ and, therefore, that $tr(\Gamma) \vdash^{*} PUx \parallel y$; but the version of $PUx \parallel y$ is Oxy.

THEOREM 10.16. Given the conditions of Theorem 10.15: if $tr(\Gamma), tr(Ixy) \vdash^{*} a = a$, then $tr(\Gamma) \vdash^{*} x \parallel y$.

Proof. If $tr(\Gamma), tr(Ixy) \vdash^{*} a = a$ we show, by analogy with the proof of Theorem 10.15, that $Ixy \notin \Gamma$, that there exists a negative element γ in $tr(\Gamma)$ for which $tr(\Gamma \cup \{Ixy\})^{-}, \gamma \vdash^{*} a = a$ and that there are functors R and S for which, if γ is $z \parallel w$, then $tr(\Gamma \cup \{Ixy\})^{+} \vdash^{*} Rz = Sw$, and if γ is $Pz \parallel w$, then $tr(\Gamma \cup \{Ixy\})^{+} \vdash^{*} RPz = Sw$. Thus there exists a strictly $tr(\Gamma \cup \{Ixy\})$-regular sequence $b_1, b_2, \ldots, b_n$ such that b_1 is Rz (if γ is $z \parallel w$) or b_1 is RPz (if γ is $Pz \parallel w$), b_n is Sw and some j $(1 \leq j < n)$ is such that the origin of $b_j = b_{j+1}$ is $tr(Ixy)$ or a converse of $tr(Ixy)$.

By Theorems 10.2 and 10.3 there exists just one j which satisfies these conditions, because tr(Ixy) is $Qx = Ty$, for simple functors Q and T that do not occur in any element of tr(Γ). Since $tr(\Gamma)^{*} \overset{*}{\vdash} b_i = b_j$ and b_j is UQx or b_j is UTy, for some functor U, we deduce that γ cannot be $Pz \parallel y$; otherwise, we would have that $tr(\Gamma) \overset{*}{\vdash} RPz = UQx$ or $tr(\Gamma)^{*} \overset{*}{\vdash} RPz = UTy$, where P, Q and T are distinct simple functors, with P different from Q, which do not occur in any element of $tr(\Gamma)^{*}$, and this is absurd, by Theorems 5.3 and 5.4. Therefore, γ has the form $z \parallel w$ and b_1 is Rz. We then have two alternatives:

(a) The origin of $b_j = b_{j+1}$ is $Qx = Ty$: in this case, we have $tr(\Gamma)^{*} \overset{*}{\vdash} Rz = UQx$ and $tr(\Gamma)^{*} \overset{*}{\vdash} VTy = Sw$, for some functors U and V. By Theorems 5.3 and 5.4, we have that, for functors M and N, $tr(\Gamma)^{*} \overset{*}{\vdash} x = Mz$ and $tr(\Gamma)^{*} \overset{*}{\vdash} y = Nw$; but $tr(\Gamma) \overset{*}{\vdash} Mz \parallel UNw$, and $tr(\Gamma)^{*} \overset{*}{\vdash} x \parallel y$.

(b) The origin of $b_j = b_{j+1}$ is $Ty = Qx$: the proof is similar to that of the preceding case.

THEOREM 10.17. $\Gamma \cup \{\beta\}$ is a set of formulas of $\mathcal{L}_1$. $\Gamma \overset{*}{\vdash} \beta$ if, and only if, either there exists a term a of $\mathcal{L}$ such that $tr(\Gamma) \overset{*}{\vdash} a \parallel a$ or there exists a formula α of $\mathcal{L}$ such that β is a version of α and $tr(\Gamma) \overset{*}{\vdash} \alpha$.

THEOREM 10.18. Given the hypothesis of Theorem 10.17: $\Gamma \vdash \beta$ if, and only if, there exists a term a of $\mathcal{L}$ such that $tr(\Gamma) \overset{*}{\vdash} a \parallel a$ or there exists a formula α of $\mathcal{L}$ such that β is a version of α and $tr(\Gamma) \overset{*}{\vdash} \alpha$.

Proof. Taking into account Theorems 8.1 and 10.17, we only have to show the 'only if' part of the present theorem. If $\Gamma \vdash \beta$, there exists a finite subset Δ of Γ such that the pair $\langle \beta, \Delta \rangle$ has a proof in the indirect system. If we show that $tr(\Delta) \overset{*}{\vdash} a \| a$, for some a of $\mathcal{L}$, or $tr(\Delta) \overset{*}{\vdash} \alpha$ for some α of $\mathcal{L}$ such that β is a version of α, the proof is complete because $tr(\Delta) \subseteq tr(\Gamma)$.

We prove, by induction on a proof of $\langle \beta, \Delta \rangle$ in the indirect system, that $tr(\Delta) \overset{*}{\vdash} a \| a$ for some a of $\mathcal{L}$, or $tr(\Delta) \overset{*}{\vdash} \alpha$ for some α such that β is a version of α. The sole non-trivial case of the induction is that in which we suppose that the pair $\langle \beta, \Delta \rangle$ is obtained from previous pairs $\langle \gamma, \Delta \cup \{C(\beta)\} \rangle$ and $\langle C(\gamma), \Delta \cup \{C(\beta)\} \rangle$. By the induction hypothesis either there is an a such that $tr(\Delta), tr(C(\beta)) \overset{*}{\vdash} a \| a$ or there exists a formula μ and a formula π of $\mathcal{L}$ such that: γ is a version of μ, $C(\gamma)$ is a version of π, $tr(\Delta), tr(C(\beta)) \overset{*}{\vdash} \mu$, and $tr(\Delta), tr(C(\beta)) \overset{*}{\vdash} \pi$. We may suppose that γ is positive. There are then two subcases: (i) μ is of the form $x = Qy$ and π is of the form $Px \| y$, where P and Q are non-empty functors; (ii) μ has the form $Px = Qy$ and π has the form $x \| y$, where P and Q are non-empty functors. In both subcases, we have that $tr(\Delta), tr(C(\beta)) \overset{*}{\vdash} Px \| Px$. So, in any case there is an a such that $tr(\Delta), tr(C(\beta)) \overset{*}{\vdash} a \| a$. If $C(\beta)$ is positive, then $tr(\Delta) \overset{*}{\vdash} \alpha$, for some α such that β is a version of α, by Theorems 10.15 and 10.16. Let $C(\beta)$ then be negative: if Δ is inconsistent*, then our final conclusion follows from Theorem 10.14. Let us suppose that Δ

is consistent*: we conclude that $C(\beta) \notin \Delta$, because otherwise we would have that $tr(\Delta) \overset{*}{\vdash} Px \parallel Px$ and this is incompatible with the consistency* of Δ, by Theorem 10.14. But Theorem 5.2 assures us that there exists a negative formula φ such that $\varphi \in tr(\Delta) \cup \{tr(C(\beta))\}$ and $tr(\Delta)^*, \varphi \overset{*}{\vdash} Px \parallel Px$. If $\varphi \in tr(\Delta)$, then we would have that $tr(\Delta) \overset{*}{\vdash} Px \parallel Px$, and this is impossible. Thus, $tr(\Delta)^*, tr(C(\beta)) \overset{*}{\vdash} Px \parallel Px$. By Theorems 10.12 and 10.13, there exists a formula α of $\mathcal{L}$ such that β is a version of α and $tr(\Delta) \vdash \alpha$.

THEOREM 10.19. Let $\Gamma \cup \{\beta\}$ be a set of formulas of $\mathcal{L}_1$. $\Gamma \vdash \beta$ if, and only if, $\Gamma \overset{*}{\vdash} \beta$.

Proof. By Theorems 10.17 and 10.18.

COROLLARY 1. Γ is consistent if, and only if, Γ is consistent*.

COROLLARY 2. $\Gamma \overset{*}{\vdash} \beta$ if, and only if, Γ implies β.

Every simple functor of $\mathcal{L}$ has the form P_j or the form Q_j, for some numerical subscript j. For each simple functor of form P_j and each interpretation $\mathfrak{I}$ of $\mathcal{L}_1$, we define a function $\mathfrak{I}^{P_j}$, whose domain is the set of the non-empty subsets of the set $\cup\{h$: there exists a simple term x such that $h = \mathfrak{I}(x)\}$. Let us call this last set $\theta^{\mathfrak{I}}$. We then define $\mathfrak{I}^{P_j}$ by the following clauses: 1) if there are simple terms x and y such that Oxy is the j-th formula of $\mathcal{L}_1$, $\mathfrak{I}(x) \nsubseteq \mathfrak{I}(y)$, and $k = \mathfrak{I}(x)$, then $\mathfrak{I}^{P_j}(k) = \mathfrak{I}(x) - \mathfrak{I}(y)$; 2) if there exist simple terms x and y such that Ixy is the j-th formula of $\mathcal{L}_1$, $\mathfrak{I}(x) \cap \mathfrak{I}(y) \neq \phi$, and $k = \mathfrak{I}(x)$, then $\mathfrak{I}^{P_j}(k) = \mathfrak{I}(x) \cap \mathfrak{I}(y)$; 3) otherwise, $\mathfrak{I}^{P_j}(k) = k$.

For each simple functor of form Q_j and for each interpretation $\mathfrak{I}$ of $\mathcal{L}_1$, we define a function $\mathfrak{I}^{Q_j}$, whose domain is also the set on the non-empty subsets of $\theta^{\mathfrak{I}}$, by the clauses: 1) if there exist simple terms x and y such that Axy is the j-th formula of $\mathcal{L}_1$, $\mathfrak{I}(x) \subseteq \mathfrak{I}(y)$, and $k = \mathfrak{I}(y)$, then $\mathfrak{I}^{Q_j}(k) = \mathfrak{I}(x)$; 2) if there are simple terms x and y such that Ixy is the j-th formula of $\mathcal{L}_1$, $\mathfrak{I}(x) \cap \mathfrak{I}(y) \neq \phi$, and $k = \mathfrak{I}(y)$, then $\mathfrak{I}^{Q_j}(k) = \mathfrak{I}(x) \cap \mathfrak{I}(y)$; 3) otherwise, $\mathfrak{I}^{Q_j}(k) = k$.

Let $\mathfrak{I}$ be any interpretation of $\mathcal{L}_1$. For all natural numbers j, $\mathfrak{I}^{P_j}$ and $\mathfrak{I}^{Q_j}$ are restrictive functions on $\theta^{\mathfrak{I}}$. Now, consider the ordered pair whose first coordinate is $\theta^{\mathfrak{I}}$ and whose second coordinate is the function which associates the set $\mathfrak{I}(x)$ with each simple term x of $\mathcal{L}$, and the function $\mathfrak{I}^R$ with each simple functor R of $\mathcal{L}$. Such a pair is clearly an interpretation of $\mathcal{L}$, which we shall denote by $I^{\mathfrak{I}}$.

THEOREM 10.20. Γ is a set of formulas of $\mathcal{L}_1$ and $\mathfrak{I}$ is an interpretation of $\mathcal{L}_1$. If $\mathfrak{I}$ is a model of Γ, then $I^{\mathfrak{I}}$ is a model of $tr(\Gamma)$.

Proof. Let $\mathfrak{I}$ be a model of Γ. Let β be an element of $tr(\Gamma)$ and α the element of Γ such that β is $tr(\alpha)$. We have four cases:

(i) α is Axy, β is $x = Qy$: we have $\mathfrak{I}(x) \subseteq \mathfrak{I}(y)$ and Axy is the j-th formula of $\mathcal{L}_1$. Thus, $\mathfrak{I}^{Q_j}(\mathfrak{I}(y)) = \mathfrak{I}(x)$, i.e., $I^{\mathfrak{I}}(Q_j y) = I^{\mathfrak{I}}(x)$, and $I^{\mathfrak{I}}(\beta) = 1$.

(ii) α is Ixy, β is $P_j x = Q_j y$: we have that $\mathfrak{I}(x) \cap \mathfrak{I}(y) \neq \phi$ and Ixy is the j-th formula of $\mathcal{L}_1$. Therefore, $\mathfrak{I}^{Q_j}(\mathfrak{I}(y)) = \mathfrak{I}(x) \cap \mathfrak{I}(y) = \mathfrak{I}^{P_j}(\mathfrak{I}(x))$. Consequently, $I^{\mathfrak{I}}(P_j x) = I^{\mathfrak{I}}(Q_j y)$, and $I^{\mathfrak{I}}(\beta) = 1$.

(iii) α is Oxy, β is $P_j x \parallel y$: $\mathfrak{I}(x) \not\subseteq \mathfrak{I}(y)$ and Oxy is the j-th formula of $\mathcal{L}_1$. Thus, $\mathfrak{I}^{P_j}(\mathfrak{I}(x)) = \mathfrak{I}(x) - \mathfrak{I}(y)$, i.e., $I^{\mathfrak{I}}(P_j x) = \mathfrak{I}(x) - \mathfrak{I}(y)$, and $I^{\mathfrak{I}}(P_j x) \cap I^{\mathfrak{I}}(y) = \varphi$. Therefore, $I^{\mathfrak{I}}(\beta) = 1$.

(iv) α is Exy, β is $x \parallel y$: $\mathfrak{I}(x) \cap \mathfrak{I}(y) = \phi$, i.e., $I^{\mathfrak{I}}(x) \cap I^{\mathfrak{I}}(y) = \phi$, and $I^{\mathfrak{I}}(\beta) = 1$.

THEOREM 10.21. Γ is a set of formulas of $\mathcal{L}_1$; β is a formula of $\mathcal{L}_1$, and α is a formula of $\mathcal{L}$, such that β is a version of α. If $tr(\Gamma)$ implies α, then Γ implies β.

Proof. Let $\mathfrak{I}$ be a model of Γ. By Theorem 10.20, $I^{\mathfrak{I}}$ is a model of $tr(\Gamma)$ and, then, $I^{\mathfrak{I}}(\alpha) = 1$. We have four cases: (i) β is Axy, α is of the form $x = Ry$: by the definition of an interpretation of $\mathcal{L}$, $I^{\mathfrak{I}}(x) \subseteq I^{\mathfrak{I}}(y)$; that is, $\mathfrak{I}(x) \subseteq \mathfrak{I}(y)$ and $\mathfrak{I}(\beta) = 1$.

(ii) β is Ixy, α is of the form $Sx = Ry$: we have that $I^{\mathfrak{I}}(x) \cap I^{\mathfrak{I}}(y) \neq \phi$; that is, $\mathfrak{I}(x) \cap \mathfrak{I}(y) \neq \phi$ and $\mathfrak{I}(\beta) = 1$.

The other cases can be similarly treated.

THEOREM 10.22. Let $\Gamma \cup \{\beta\}$ be a set of formulas of $\mathcal{L}_1$. Γ implies β if, and only if, there exists a formula α of $\mathcal{L}$ such that β is a version of α and $tr(\Gamma)$ implies α.

Proof. Given Theorem 10.21, it only remains for us to prove the only-if-part. If Γ implies β, then by Theorem 10.19, $\Gamma \overset{*}{\vdash} \beta$. By Theorem 10.18, we have two possibilities: (i) there exists a formula α of $\mathcal{L}$ such that β is a version of α and $tr(\Gamma) \overset{*}{\vdash} \alpha$; in this hypothesis, $tr(\Gamma)$ implies α, by the completeness theorem for the strong algebraic system, because any deduction of the weak algebraic system is obviously also a deduction of the strong system; (ii) there exists a term a of $\mathcal{L}$ such that $tr(\Gamma) \overset{*}{\vdash} a \parallel a$; thus, $tr(\Gamma) \vdash a \parallel a$ and, by the rule of trivialization, $tr(\Gamma) \vdash \alpha$, for any formula α of $\mathcal{L}$, and $tr(\Gamma)$ vacuously implies α, for any formula α of $\mathcal{L}$. Hence, $tr(\Gamma)$ implies $tr(\beta)$, where β is a version of $tr(\beta)$.

THEOREM 10.23. Given the conditions of Theorem 10.22: $\Gamma \vdash \beta$ if, and only if, there exists a formula α of $\mathcal{L}$ such that β is a version of α and $tr(\Gamma) \vdash \alpha$.

Proof. By Theorem 10.22 and the completeness theorems for the Aristotelian system and strong algebraic system.

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